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DETERMINATION OF THE EQUILIBRIUM BOUNDARIES OF A TWO-LAYER SYSTEM CONVECTION STABILITY

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The method which permits the determination of a large number of critical Rayleigh numbers and respective critical motions is applied to the determination of convection stability boundaries of a two-layer system. The method consists essentially of reducing the problem to the algebraic problem of eigenvalues by discretizing the equations by the method of finite elements or finite differences. Only the scheme of the method based on discretization by finite elements is then presented.

Galerkin's method [1] is generally used for determining critical Rayleigh numbers and related critical motions in a closed cavity. As a rule, the results are valid only for the determination of several lower layers of the spectrum and, when passing to a cavity of different shape it is necessary to use the system of basis functions.

Application of the proposed method to the determination of the spectrum of critical Rayleigh numbers and critical motions of a two-layer system does not present great difficulties, and is demonstrated here on the example of a square-cross section cavity filled with two immiscible fluids in equal proportions. It is assumed that the two fluid interface is horizontal but not subject to deformations (high surface tension) and lies in the cavity middle. Thermo capillary effects are disregarded. Below, we consider two cases: a cavity with solid perfectly heat-conducting walls, and a convection cell of an infinite horizontal fluid layer (solid horizontal boundaries and free side boundaries).

1. The use of this method requires that the problem be formulated in variations. We obtain the respective functional by formulating the equations for neutral plane perturbations of equilibrium of the incompressible fluid in terms of the stream function ψ and vorticity φ, and the perturbation of the equilibrium temperature T

$$\Delta\psi - \varphi = 0, \quad \Delta\varphi - \sqrt{R}\frac{\partial T}{\partial x} = 0, \quad \Delta T - \sqrt{R}\frac{\partial \psi}{\partial x} = 0 \tag{1.1}$$

$$(R = g\beta A l^4 / (\nu\chi), \ \Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2)$$

Parameters l, ν / l and $(\nu / l)[A\nu / (\beta g\chi)]^{1/2}$, where l is a characteristic dimension, A the temperature gradient, and the remaining notation conventional, are chosen as units of length, velocity, and temperature, respectively.

Solutions of Eqs. (1.1) yield the extremum of functional

$$J = \int\limits_S \left\{ \frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{1}{2}\left[\varphi^2 - \left(\frac{\partial T}{\partial x}\right)^2 - \right.\right. \tag{1.2}$$

$$\left(\frac{\partial T}{\partial y}\right)^2 + \sqrt{R}\left(\psi\frac{\partial T}{\partial x} - T\frac{\partial\psi}{\partial x}\right)\Bigg]\Bigg\} dx\,dy$$

with boundary conditions for ψ and φ that correspond to free or solid boundaries, and the conditions for T relevant to thermally insulated or perfectly heat-conducting boundaries.

The functional for a two-layer system is the sum of functionals of the form (1.2), since the reduction to dimensionless form with respect to parameters of one of the fluids yields different coefficients of equations for the upper and lower fluids

$$\Delta\psi^{(k)} - \varphi^{(k)} = 0 \tag{1.3}$$

$$\Delta\varphi^{(k)} - \sqrt{R}\, a^{(k)}\frac{\partial T^{(k)}}{\partial x} = 0$$

$$\Delta T^{(k)} - \sqrt{R}\, b^{(k)}\frac{\partial\psi^{(k)}}{\partial x} = 0, \quad k = 1, 2$$

$$a^{(1)} = 1, \quad a^{(2)} = \varepsilon, \quad b^{(1)} = 1, \quad b^{(2)} = \chi_r\varkappa_r$$

in which, here and subsequently, indices 1 and 2 denote quantities related to the upper and lower fluid layers, respectively. For the units of length, velocity, and temperature in (1.3) we use, respectively, the half-length of the square side l, ν_1 / l, and $(\nu_1 / l)\,[A_1\nu_1 / (\beta_1 g\chi_1)]^{1/2}$, where A_1 represents the equilibrium temperature gradient of upper layer.

Equation (1.3) contains four dimensionless parameters: the Rayleigh number $R = g\beta_1 l^4 A_1 / (\nu_1\chi_1)$ and the ratio of the upper and lower fluid [layer] characteristics $\varkappa_r = \varkappa_1 / \varkappa_2$, $\chi_r = \chi_1 / \chi_2$, $\varepsilon = \nu_1\beta_2 / (\nu_2\beta_1)$.

These equations must be complemented by conditions at the fluid interface ($y = 1$). Absence of vertical displacements of the interface and the equality of horizontal velocities and temperatures imply that

$$\psi^{(1)} = \psi^{(2)} = 0, \quad \frac{\partial\psi^{(1)}}{\partial y} = \frac{\partial\psi^{(2)}}{\partial y}, \quad T^{(1)} = T^{(2)} \tag{1.4}$$

The continuity of shearing stresses and heat fluxes is defined by formulas

$$\eta_r\varphi^{(1)} = \varphi^{(2)}, \quad \varkappa_r\frac{\partial T^{(1)}}{\partial y} = \frac{\partial T^{(2)}}{\partial y} \tag{1.5}$$

The conditions at the interface contain one more parameter η_r which is the ratio of the upper and lower fluid coefficients of dynamic viscosities.

Along the solid perfectly heat conducting horizontal boundaries of the cavity the following relations are satisfied:

$$\psi = \frac{\partial\psi}{\partial y} = 0, \quad T = 0 \quad (y = 0, 2) \tag{1.6}$$

and the conditions at the perfectly heat conducting solid side boundaries are of the form

$$\psi = \frac{\partial\psi}{\partial x} = 0, \quad T = 0 \quad (x = 0, 2) \tag{1.7}$$

In the case of free boundaries (the side boundaries of the convection cell) these conditions are of the form

$$\psi = \varphi = 0, \quad \frac{\partial T}{\partial x} = 0 \quad (x = 0, 2) \tag{1.8}$$

The complete functional, whose extremum is obtained by solving Eqs. (1.3) with boundary conditions (1.4) – (1.8), is of the form

$$J = J^{(1)} + \frac{\chi_r}{\varepsilon} J^{(2)} + \left(1 - \frac{\eta_r \chi_r}{\varepsilon}\right) \int_0^2 \left(\varphi^{(1)} \frac{\partial \psi^{(1)}}{\partial y}\right)_{y=1} dx \tag{1.9}$$

$$J^{(k)} = \int_{S_k} \left\{ \frac{\partial \psi^{(k)}}{\partial x} \frac{\partial \varphi^{(k)}}{\partial x} + \frac{\partial \psi^{(k)}}{\partial y} \frac{\partial \varphi^{(k)}}{\partial y} + \frac{1}{2} (\varphi^{(k)})^2 - \frac{1}{2} \frac{a^{(k)}}{b^{(k)}} \left[\left(\frac{\partial T^{(k)}}{\partial x}\right)^2 + \left(\frac{\partial T^{(k)}}{\partial y}\right)^2 \right] + \frac{1}{2} \sqrt{R}\, a^{(k)} \left(\psi^{(k)} \frac{\partial T^{(k)}}{\partial x} - T^{(k)} \frac{\partial \psi^{(k)}}{\partial x}\right) \right\} dx\, dy$$

where integration is carried out over the part of the cavity occupied by the respective fluid, and was constructed by the authors.

2. Let us reduce the problem to the algebraic one of eigenvalues. In conformity with the concept of the finite elements method we decompose the [cavity] cross section in a certain number of triangular elements. The most convenient form of decomposition is that in which one of the sides of every element lies on the interface (Fig. 1). This allows to represent integrals $J^{(1)}$ and $J^{(2)}$ in (1.9) as the sum of integrals taken over individual elements, and the integral along the interface as the sum of integrals over individual segments.

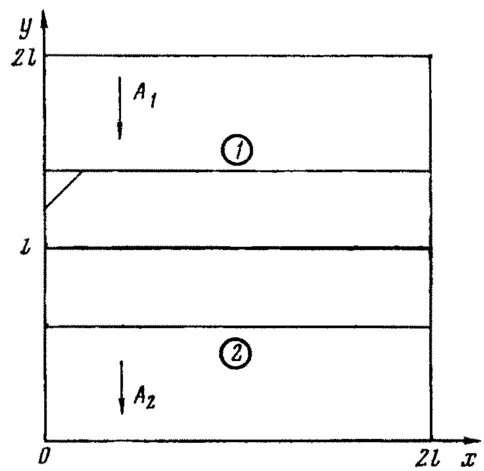


Fig. 1

By approximating functions ψ, φ, and T inside each element (in both fluids) by first power polynomials in x and y it is possible to represent functional J in terms of values of ψ, φ, and T at nodal points coinciding with vertices of elements. Variation of J with respect to these values yields a system of linear algebraic equations in which the unknowns are: values of the stream function ψ at inner points of the region (except the interface where $\psi = 0$), the values of vorticity φ at inner points, at rigid boundaries, and at the interface, and the values of temperature T at inner points, on heat insulated boundaries and at the interface [of fluids]. By eliminating φ and T from the unknowns the system is reduced to a system of equations in ψ at N inner nodes.

To simplify the construction of the matrix of this system it is convenient to express the vorticity and temperature at the interface and boundaries of the cavity entirely in terms of variables at inner points. For nodes lying at the interface this can be achieved, for example, in the following manner.

We expand variables in border nodes in Taylor series

$$T_1' = T_0 + \left(\frac{\partial T_1}{\partial y}\right)_{y=1} h_1, \quad T_2' = T_0 - \left(\frac{\partial T_2}{\partial y}\right)_{y=1} h_2 \tag{2.1}$$

$$\psi_1' = \left(\frac{\partial \psi_1}{\partial y}\right)_{y=1} h_1 + \frac{1}{2}\left(\frac{\partial^2 \psi_1}{\partial y^2}\right)_{y=1} h_1^2, \tag{2.2}$$

$$\psi_2' = -\left(\frac{\partial \psi_2}{\partial y}\right)_{y=1} h_2 + \frac{1}{2}\left(\frac{\partial^2 \psi_2}{\partial y^2}\right)_{y=1} h_2^2$$

where h_1 is the distance to the nearest border node in the upper fluid and h_2 that to the nearest node in the lower fluid; the zero subscript denotes quantities at the interface and the prime at border nodes.

With the second formula of (1.5) taken into account, from (2.1) we obtain the following formula for T_0 in terms of values at border nodes:

$$T_0 = \left(\varkappa_r T_1' + \frac{h_1}{h_2} T_2'\right)\left(\varkappa_r + \frac{h_1}{h_2}\right)^{-1} \tag{2.3}$$

and, taking into account the second of conditions (1.4), the first of conditions (1.5), and the first of Eqs. (1.3), from (2.2) we obtain for vorticities $\varphi_0^{(1)}$ and $\varphi_0^{(2)}$ at the interface the formulas

$$\varphi_0^{(1)} = 2\,\frac{\psi_1'/h_1 + \psi_2'/h_2}{\eta_r h_2 + h_1}, \quad \varphi_0^{(2)} = \eta_r \varphi_0^{(1)} \tag{2.4}$$

(their values for the first and second fluid are different).

Formulas (2.3) and (2.4) enable us to reduce the input system to one of $3N$ algebraic equations.

For convenience of exposition we write this system in matrix form. We introduce column vectors $\{\psi\}$, $\{\varphi\}$, and $\{T\}$ whose elements represent the values of stream and vorticity functions, and of temperature at the nodes. We also introduce matrices $(\psi\varphi)$, (ψT), $(\varphi\varphi)$, $(\varphi\psi)$, (TT), $(T\psi)$ of order N which make it possible to represent the system of algebraic equations in the form of three matrix equations each of which is obtained by varying the functional with respect to ψ, φ, and T

$$(\psi\varphi)\{\varphi\} + \sqrt{R}\,(\psi T)\{T\} = 0$$
$$(\varphi\varphi)\{\varphi\} + (\varphi\psi)\{\psi\} = 0$$
$$(TT)\{T\} + \sqrt{R}\,(T\psi)\{\psi\} = 0$$

Eliminating $\{\varphi\}$ and $\{T\}$ from these equations we obtain for the values of ψ in nodes the system of equations

$$[(M) - R^{-1}(E)]\{\psi\} = 0 \tag{2.5}$$

where (E) is a unit matrix, and the N order matrix (M) is related to the matrices defined above as follows:

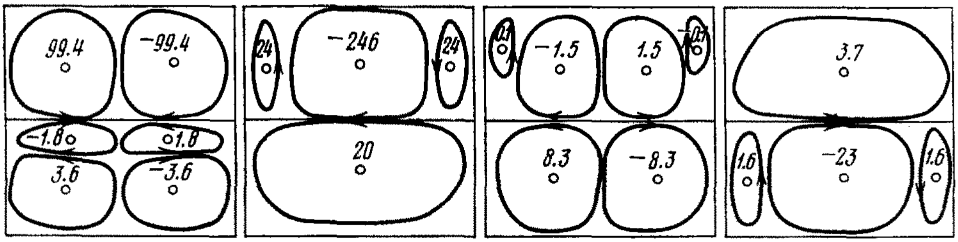
$$(M) = -(\varphi\psi)^{-1}(\varphi\varphi)(\psi\varphi)^{-1}(\psi T)(TT)^{-1}(T\psi) \tag{2.6}$$

Equations (2.5) represent a problem in eigenvalues, where the latter are quantities inverse of the critical Rayleigh numbers, while the eigenvectors represent the set of values of ψ that correspond to critical motions.

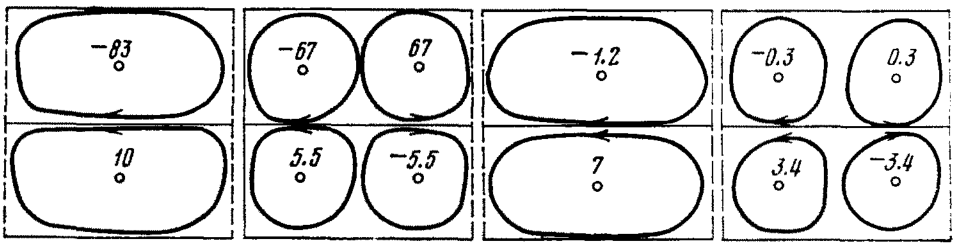
Note that the matrices used in (2.6) can be calculated by the same procedure for cavities of any arbitrary form, with the necessary alterations of coordinates of the element nodal points.

The eigenvalues and eigenvectors of system (2.5) are conveniently determined by the power method [2] beginning with the highest absolute value (the lowest critical Rayleigh number). The eigenvalues presented below were calculated by the method of reduction [2].

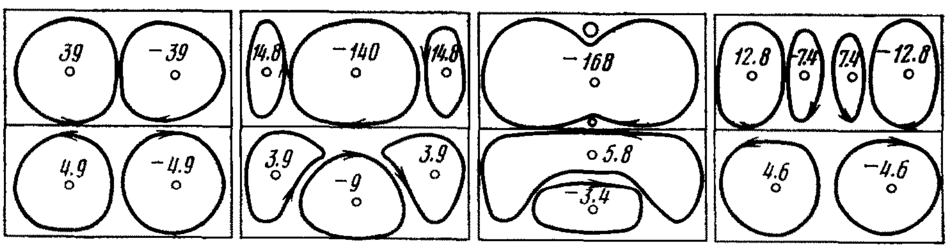
3. Let us consider some of the computation results. The lower four critical motions for $\varkappa_r = \chi_r = 1$, $\varepsilon = \eta_r = 0.5$ and Rayleigh numbers 27530 (a), 28710 (b), 51640 (c), 51820 (d) in a cavity with solid boundaries are shown in Fig. 2,



a b Fig. 2 c d



a b Fig. 3 c d



a b c d

Fig. 4

and in the convection cell of an infinite horizontal fluid layer for Rayleigh numbers 13670 (*a*), 16480 (b), 23050 (c), 30280 (d) they appear in Fig. 3. The figures at the center of vortices indicate the maximum absolute values of the streamfunction of the particular vortex (vorticity).

With the above values of parameters the conditions of convection onset are different for the upper and lower [layer] fluids. If Rayleigh numbers determined separately by the equilibrium temperature gradients and the respective parameters of the upper and lower fluid $R_1 = g\beta_1 A_1 l^4 / (\nu_1 \chi_1)$ and $R_2 = g\beta_2 A_2 l^4 / (\nu_2 \chi_2)$, then for such parameters $R_1 = 2R_2$.

Hence, in the absence of thermal and dynamic interaction over the interface, on reaching the threshold Rayleigh number, motion would appear in the upper fluid, while the lower one would remain in equilibrium.

Owing to the interaction over the interface motion is simultaneously originated in both fluids (see Figs. 2, a and 3, a), but the intensity of motion in the upper fluid is considerably higher than in the lower. It is interesting that in a cavity with solid walls (Fig. 2, a) the motion in the lower half decomposes into a system of four vortices of which the upper two are apparently induced by stresses in the interface, and the two lower ones are due to buoyancy.

Similar structures were obtained in nonlinear computations [3], where finite amplitude convective motions in a two-layer system were numerically determined. The present investigation shows that there are no grounds for interpreting such motions as the consequence of nonlinear effects.

Examination of some of the critical motions will show the somewhat unexpected effect of the interaction over the interface. In the third and fourth critical motions the circulation intensity in the lower fluid is higher than in the upper, unlike previously (although the Rayleigh numbers are in inverse relation). The third motion is similar to the first, and the fourth to the second (this is particularly noticeable in the case of a cell with free side boundaries, where the pattern of motions is simpler, Fig. 3).

In connection with this we may again recall the nonlinear computations in [3] which had disclosed that as the temperature difference at horizontal boundaries increases in a similar situation $(R_2 < R_1)$ the finite amplitude motion in the lower fluid may exceed the intensity of motion in the upper fluid. This is apparently explained by the admixture of a third critical motion to the motion originating at the stability threshold.

Four lower critical motions are shown in Fig. 4 for the system water-mercury at a temperature of 20°C and Rayleigh numbers 32780 (*a*), 33355 (b), 89029 (c), 110732 (d). The ratio of Rayleigh numbers determined by parameters of the first and second fluids is in this case $R_1 = 63.4\ R_2$. It is seen that in this situation the motions following the threshold one have a complex pattern, while the onset of counterflows in the threshold motion pattern at the interface is prominent.

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